

Loop Correlators and Theta States in 2D Yang-Mills Theory¹

G. Grignani^(a), L. Paniak^(b), G. W. Semenoff^(b) and P. Sodano^(a)

*(a) Dipartimento di Fisica and Sezione I.N.F.N., Università di Perugia, Via A. Pascoli
I-06123 Perugia, Italy*

*(b) Department of Physics and Astronomy
University of British Columbia
6224 Agricultural Road
Vancouver, British Columbia, Canada V6T 1Z1*

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Abstract

Explicit computations of the partition function and correlation functions of Wilson and Polyakov loop operators in theta-sectors of two dimensional Yang-Mills theory on the line cylinder and torus are presented. Several observations about the correspondence of two dimensional Yang-Mills theory with unitary matrix quantum mechanics are presented. The incorporation of the theta-angle which characterizes the states of two dimensional adjoint QCD is discussed.

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1 Introduction

Two dimensional (2D) Yang-Mills theory is an interesting example of a topological field theory [1, 2, 3]. It has been used extensively as a toy model where features that it shares with higher dimensional gauge theories can be studied in a simplified context [4, 5]. As a quantum field theory, it has no propagating degrees of freedom. It nevertheless exhibits a confining potential for heavy quarks at the tree level. It is also an example of a gauge theory where (in a planar geometry) the Migdal-Makeenko equations for Wilson loops can be solved explicitly. Correlators of Wilson loops can be computed in closed form [6, 7] and the area law associated with confinement physics can be demonstrated explicitly. Moreover, it has been demonstrated that the strong coupling, large N limit of 2D Yang-Mills theory on the sphere can be rewritten as a random surface model [8, 9]. This supports the long-standing conjecture that the infrared limit of a confining gauge theory is related to string theory. Its dynamics are also known to be intimately related with $D=1$ unitary matrix models which are related to two dimensional string theories [10, 11, 12].

As a topological field theory, 2D Yang-Mills theory gives a field theoretical example of localization formulae [1, 2]. It is a quantum field theory whose partition function is given exactly by evaluating the Euclidean functional integral in the saddle point approximation. The technique which is used to compute the functional integral is the so-called diagonalization method where gauge invariance is used to diagonalize the matrix-valued degrees of freedom to yield a simple, solvable model for the eigenvalues. The spectrum which is obtained this way differs from the conventional one. This difference is related to the existence of inequivalent quantizations of the gauge theory, depending on the order in which the constraints are imposed and quantization is done [13, 14, 15, 16], and was noted by Witten in his analysis of 2D Yang-Mills theory [1]. The diagonalization procedure has also been used to compute the partition function and correlators of Polyakov loop operators in ref. [17, 18].

In this Paper we shall summarize some of our observations about 2D Yang-Mills theory. We shall begin by making a few remarks about the correspondence between Yang-Mills theory and unitary matrix quantum mechanics. We shall make extensive use of one such well-known correspondence [19, 10, 20, 21] where the unitary matrices are the Wilson loops. We shall also formulate another one where the dynamical variables are unitary matrices related to Polyakov loops. These matrices determine the gauge group holonomy that the wave-functions of heavy quarks accumulate in the quantized Yang-Mills theory. The latter correspondence is related to recent work where it was shown that the question of whether or not a gauge theory with adjoint matter and at finite temperature and density exists in a confined or de-confined phase could be related to the question of symmetry breaking in a certain nonlinear sigma model [17, 22, 25]. In that model, the dynamical variable is a unitary matrix whose trace coincides with the Polyakov loop operator. The Polyakov loop operator is an order parameter for breaking the discrete symmetry and its expectation value characterizes confinement in finite temperature adjoint gauge theories [26, 27]. In general the action of the non-linear sigma model in question is a complicated effective action coming from integrating gauge and matter fields from the functional integral [28, 29]. However, it was argued in ref. [17] that the effective action has a simple form in 2D Yang-Mills theory and the resulting model could be solved exactly using the diagonalization method [30]. This idea was generalized to Yang-Mills theory with external sources in [31] where it was used to show that there is a de-confining phase transition at infinite N limit of a hot and dense gas of heavy quarks.

In this Paper, we shall complete the program begun in ref. [17] of presenting a full computation of the correlators of loop operators in 2D Yang-Mills theory using the diagonalization method. We are particularly interested in the sectors of the theory with non-trivial theta vacua. We will present an explicit computation of the two-point correlators of Wilson and Polyakov loop operators in a theta-sector of 2D Yang-Mills theory at finite temperature. We shall consider both the case where the space is a circle, so that the spacetime is a torus, or the space is a line with fixed boundary conditions, so that the spacetime is a cylinder. We shall compute the partition function, find the behavior of loop correlators and deduce the quark-antiquark potential for both of these cases. We show that, in the limit of infinite volume, our result reproduces the already known facts [32, 33, 34] that, in a sector of the theory with non-zero theta angle, fundamental representation quarks have a repulsive interaction and adjoint quarks have a screened, short-ranged interaction. Our results for the correlators of Polyakov loop operators are different from those found in previous computations of these quantities using the diagonalization technique [17, 18].

Our principal result is the demonstration that the diagonalization technique can be modified in such a way that it obtains the conventional spectrum of 2D Yang-Mills theory on a circle. Furthermore, we demonstrate that the two-point correlators which are obtained by this method agree in various limits with those obtained by the group theoretical character expansion technique. The distinct advantage of the diagonalization method is that it can be used to obtain explicit formulae for correlators of higher winding loops.

1.1 Theta states

It was first argued by Witten [32] that 2D Yang-Mills theory, or 2D QCD with adjoint matter, has theta states. Since the Yang-Mills fields transform in the adjoint representation of the gauge group G , the true gauge group is the factor group of G by its center, C , G/C . Generally, the resulting group is not simply connected. For example, if G itself is simply connected ($\Pi_1(G) = 0$), which we shall henceforth assume, then

$$\Pi_1(G/C) = \Pi_0(C) = C \quad (1.1)$$

This has non-trivial consequences for any gauge theory that is defined on a spacetime which itself has non-trivial fundamental group. For the moment, let us assume that the space is a circle, S^1 .² The group of time-independent gauge transformations, \mathcal{G} , which is formed from the smooth mappings of the circle, S_1 , to G/C has the property

$$\Pi_0(\mathcal{G}) = \Pi_1(G/C) = C \quad (1.2)$$

Since the Hamiltonian is gauge invariant, and the quantum states are invariant under gauge transformations in the component of the gauge group \mathcal{G} which contains the identity, the quantum states must carry an irreducible unitary representation of the Abelian group C . All such representations are one-dimensional. Each distinct representation is called a theta-state (this terminology derives from the representations of elements of $U(1)$ by phases $e^{i\theta}$).

² We could as well use the real line R^1 . In the case of the real line, the gauge group is taken as the set of those gauge transformations which go to the identity at infinity.

All gauge invariant operators have vanishing matrix elements between states which carry different representations of C . Thus, there is a superselection rule which allows one to choose one particular representation to characterize all of the quantum states of the theory.

For example, if $G = SU(N)$ then its center is $C = Z_N$, the cyclic group of order N and

$$\Pi_0(\mathcal{G}) = \Pi_1(SU(N)/Z_N) = Z_N \quad (1.3)$$

There are N irreducible unitary representations of the center, Z_N , which have elements

$$1, e^{i\theta}, e^{2i\theta}, \dots, e^{i(N-1)\theta}$$

where the N allowed values of θ are

$$\theta = 0, 2\pi/N, \dots, 2\pi k/N, \dots, 2\pi(N-1)/N$$

The partition function of Yang-Mills theory is

$$Z = \int dA_\mu \exp \left(- \int_M d^2x \frac{1}{2e^2} \text{Tr}(F_{\mu\nu}^2) \right) \quad (1.4)$$

Let us consider the case where the Euclidean space-time is a Riemann surface, M . The gauge fields belong to topological sectors. In a generic topological sector, the connection one-form A is not globally defined, but is a smooth 1-form only in coordinate patches. Consider a triangulation of the space M by a set of simply connected subspaces T^I labeled by I ($M = \cup_I T^I$) and let the connection in patch I be A^I . At the boundary between patches I and J , the connections are related by a gauge transformation $A^I = (A^J)^{g^{IJ}}$.

We can change a gauge field which is in one topological sector to one in another sector by the following surgical procedure [1]: Consider a particular field configuration A_I in patch I and transition functions g_{IJ} which relates A_I to the connections in neighboring patches. We can change topological sectors by replacing A_I by its gauge transform A^g and the transition functions g_{IJ} by $g_{IJ}g$ if we choose g to have the property that the restriction of g to the boundary of the patch (which is isomorphic to S^1) is an element of \mathcal{G} , the smooth mappings of S^1 to G/C , which is in a non-trivial component of $\Pi_0(\mathcal{G}) = C$. The resulting field configuration will be in a different topological sector from the original one. The number of topological sectors is equal to the number of elements in C . Summing over the topological sectors in the path integral with complex unimodular weights given by the corresponding elements of the irreducible unitary representation of the center, C , produces the partition function in a θ sector.

The above arguments depend on certain smoothness requirements for the class of field configurations which contribute to the path integral and also of the gauge transformations which are allowed in the Hamiltonian formalism. However, as independent evidence for the existence of theta states, they have been given a physical interpretation similar to the theta-vacuum in the Schwinger-model [35, 36] as being created by the existence of charges at the boundaries of the space. The theta parameter is known to affect the spectrum of mesons in 2D QCD with heavy quarks [32, 33]. Arguments of stability of such systems have also been shown to be consistent with the topological classification of theta states [32, 34].

1.2 Overview and Summary of Results:

In Section 2, we shall review the quantization of Yang-Mills theory in 1+1-dimensions when the space is a circle. We pay particular attention to the existence of large gauge transformations and the associated appearance of θ vacua. Then, in Section 3, we derive a number of different representations of the heat kernel of 2D Yang-Mills theory. We also present an explicit calculation of the heat kernel using the diagonalization technique. Section 4 contains a computation of loop correlators on the cylinder and torus. We also obtain results for $SU(2)$ and compare them with a similar computation using character expansion techniques and find that they agree.

We summarize our results as follows:

- We obtain the partition functions and the correlators of two loop operators for 2D QCD on both the cylinder and the torus in a θ -state when $G = SU(N)$ given in formulae (3.36), (3.53), (4.8) and (4.12). This is done using the representation of the heat kernel in (1.5) below. In the process, we show that the trace of (1.5) leads to the character representation of the partition function (1.6).

It is well known[19] that the physical states of 2D Yang-Mills theory on a circle ($x \in [0, L]$) are class functions of the Wilson loop

$$\psi_{\text{phys}}(U) = \psi_{\text{phys}}(vUv^\dagger)$$

where $vv^\dagger = 1$ and

$$U(\tau) = \mathcal{P} \exp \left(i \int_0^L dx A_1(x, \tau) \right)$$

At finite temperature where the Euclidean time is periodic ($t \in [0, \tau = 1/T]$), the Polyakov loop operator is also of interest,

$$g(x) = \mathcal{P} \exp \left(i \int_0^{1/T} dt A_0(x, t) \right)$$

The heat kernel of 2D Yang-Mills theory, defined by

$$\langle \psi_2 | e^{-H\tau} | \psi_1 \rangle = \int [dU_1][dU_2] \psi_2^*(U_2) \psi_1(U_1) K[\tau; U_2, U_1]$$

so that

$$K[\tau; U_2, U_1] = \langle U_2 | e^{-H\tau} | U_1 \rangle$$

is defined as a symmetric function of the eigenvalues of U_2 and U_1 separately. It can be written in three ways: It has the well known decomposition in terms of characters $\chi_R(U) = \text{Tr}_R U$ of U in irreducible representations R ,

$$K[\tau; U_2, U_1] = \sum_R \chi_R^*(U_2) \chi_R(U_1) e^{-e^2 L \tau C_2(R)}$$

where $C_2(R)$ is the second Casimir invariant of R . In addition, we shall find that it can be presented

- as a D=1 unitary matrix model with a background gauge field,

$$K[\tau; U_2, U_1] = \int [dg(x)] \exp \left(-\frac{1}{2e^2\tau} \int_0^L dx \text{Tr} |\nabla g + iA_2 g - igA_1|^2 \right)$$

where $g(0) = g(L)$ and

$$U_1 = \mathcal{P}e^{i \int_0^L dx A_1(x,0)} , \quad U_2 = \mathcal{P}e^{i \int_0^L dx A_1(x,\tau)}$$

- as a D=1 gauge invariant matrix model

$$K[\tau; U_2, U_1] = \int [dU(t)] dA(t) \exp \left(-\frac{1}{2e^2 L} \int_0^\tau dt \text{Tr} |\dot{U} - i[A, U]|^2 \right) \cdot \delta_{\text{cl}}(U(\tau), U_2) \delta_{\text{cl}}(U(0), U_1) J(U(\tau)) J(U(0)) \quad (1.5)$$

where $J(U)$ is the Vandermonde determinant and $\delta_{\text{cl}}(U, V)$ is the conjugation invariant delta function which equates the spectra of U and V .

- The partition function of Yang-Mills theory in a θ -state on the 2-sphere with area $L\tau$ can be written as the D=1 unitary matrix model

$$Z[\text{sphere} ; \theta] = \sum_z \mathcal{Z}(z, \theta) \int [dg_z] \exp \left(-\frac{1}{2e^2\tau} \int_0^L \text{Tr} |\nabla g_z|^2 \right)$$

where $g_z(0) = zg_z(L)$, z is an element of C , the center of G , and $Z(z, \theta)$ is the element of the θ -representation of C corresponding to z .

- The partition function of Yang-Mills theory in a θ -state on the 2-torus with area $L\tau$ can be written as the gauge invariant D=1 matrix model

$$Z[\text{torus} ; \theta] = \sum_z \mathcal{Z}(z, \theta) \int [dg_z][dA] \exp \left(-\frac{1}{2e^2\tau} \int_0^L \text{Tr} |\nabla g_z - i[A, g_z]|^2 \right)$$

In the equivalent character representation the partition function of Yang-Mills theory on a Riemann surface of genus g and in a θ -sector is given by

$$Z[g, \theta] = \sum_R (\dim R)^{2-2g} \delta(R, R_\theta) \exp \left(-e^2\tau LC_2(R) \right) \quad (1.6)$$

where R_θ are those representations which have the property

$$\chi_{R_\theta}(zU) = \mathcal{Z}(z, \theta) \chi_{R_\theta}(U) \quad (1.7)$$

2 Hamiltonian formulation of Yang-Mills theory on the circle

In this Section we shall review the Hamiltonian formulation of 2D Yang-Mills theory. Canonical quantization of the action³

$$S = \int dx dt \frac{1}{2e^2} \text{Tr} \left(\dot{A}_1 - \nabla A_0 \right)^2 \quad (2.1)$$

³Note that we use an unconventional normalization of the charge e^2 .

yields dynamical variables which are the spatial component of the gauge field $A_1(x) \equiv A(x)$ and the electric field $E(x) \equiv \dot{A}(x) - \nabla A_0(x)$, both of which are matrices in the fundamental representation of the Lie algebra of the gauge group, and can be expanded in a canonical set of generators as

$$A(x) = \sum_a T^a A^a(x) \quad , \quad E(x) = \sum_a T^a E^a(x)$$

For a compact semi-simple Lie algebra,

$$\text{Tr} T^a T^b = \frac{1}{2} \delta^{ab} \quad , \quad [T^a, T^b] = i f^{abc} T^c$$

The Hamiltonian is

$$H = 2e^2 \int_0^L dx \text{Tr} (E^2(x)) \quad (2.2)$$

and the non-vanishing canonical commutation relation is

$$[A^a(x), E^b(y)] = i \delta^{ab} \delta(x - y) \quad (2.3)$$

A_0 is a Lagrange multiplier which enforces Gauss' law as the constraint is

$$\nabla E(x) + i [A(x), E(x)] \sim 0 \quad (2.4)$$

and all variables have periodic boundary conditions,

$$E(L) = E(0) \quad , \quad A(L) = A(0) \quad (2.5)$$

The gauge transformation is

$$E(x) \rightarrow E^g(x) = g(x) E(x) g^\dagger(x) \quad , \quad A(x) \rightarrow A^g(x) = g(x) (A(x) - i \nabla) g^\dagger(x) \quad (2.6)$$

The periodic boundary conditions for $A(x)$ and $E(x)$ are preserved by gauge transformations where the matrix $g(x)$ and its first derivative are periodic up to an element of the center of the gauge group,

$$g_z(L) = z g_z(0) \quad (2.7)$$

where $z \in C$. When $z = 1$, $g_1(x)$ is said to implement a “small gauge transformation” and when $z \neq 1$, $g_z(x)$ is said to implement a “large gauge transformation”.

The operator on the right-hand-side of the constraint (2.4) generates infinitesimal small gauge transformations. As a result of the constraint (2.4), the physical phase space is the set of equivalence classes of configurations $E(x)$ and $A(x)$ where fields are in the same equivalence class if they are related by a small gauge transformation (i.e. one with a periodic gauge function and with $z = 1$ in (2.7)). The constraint could be solved at the classical level by choosing representatives of the equivalence classes. This procedure has been discussed in [37, 38]. Alternatively, after finding the quantum realization of the un-constrained theory with commutator (2.3) and Hamiltonian (2.2), the constraint (2.4) could be imposed as a physical state condition which chooses a subspace of the quantum states as ‘physical states’. It implies that physical states of the theory are invariant under all gauge transformations which can be generated by iterating infinitesimal transformations, i.e. all small gauge transformations. The coset group of all gauge transformations modulo periodic ones is isomorphic to the center

of the gauge group, C . The physical states must transform under an irreducible unitary representation of this coset.

In the functional Schrödinger picture, where states are wave-functionals, $\Psi[A]$, of gauge field configurations and the electric field is realized as

$$E^a(x) \Psi[A] = \frac{1}{i} \frac{\delta}{\delta A^a(x)} \Psi[A] \quad , \quad (2.8)$$

the physical state condition,

$$(\nabla E(x) + i[A(x), E(x)]) \Psi_{\text{phys.}}[A; \theta] = 0 \quad , \quad (2.9)$$

implies that the wave-functionals of physical states transform as

$$\Psi_{\text{phys.}}[g_z(A - i\nabla)g_z^\dagger; \theta] = \mathcal{Z}(z, \theta) \Psi_{\text{phys.}}[A; \theta] \quad (2.10)$$

where $g_z(x)$ has the boundary condition in (2.7) and $\mathcal{Z}(z, \theta)$ is the representative of the center element z in the representation labeled by θ . Since the center of the gauge group is an Abelian discrete group, all irreducible unitary representations are one-dimensional and the number of inequivalent representations is equal to the order of the group. The solution of (2.10) is

$$\Psi_{\text{phys.}}[A; \theta] = \psi[U; \theta] \quad (2.11)$$

where U is the unitary matrix

$$U \equiv \mathcal{P} \exp \left(i \int_0^L dx A(x) \right) \quad (2.12)$$

with the additional constraint that the wave-function is a class function of U :

$$\psi[U; \theta] = \psi[hUh^\dagger; \theta] \quad (2.13)$$

for any h in the fundamental representation of G . This implies that the wave-function is a symmetric function of the eigenvalues of U . On the wavefunctions, we must impose the further requirement that they lie in a particular theta-sector,

$$\psi[zU; \theta] = \mathcal{Z}(z, \theta) \psi[U, \theta] \quad (2.14)$$

Let us consider the operator U_R constructed from the path-ordered phase integral with the gauge field in an arbitrary irreducible representation R of the Lie algebra,

$$U_R \equiv \mathcal{P} \exp \left(i \int_0^L dx A^a(x) T_R^a \right) \quad (2.15)$$

where T_R^a are the generators in representation R . The electric field operates on U_R as

$$E^a(x) U = \mathcal{P} \exp \left(i \int_x^L dw A^a(w) T_R^a \right) T_R^a \mathcal{P} \exp \left(i \int_0^x dw A_R^a(w) \right) \quad (2.16)$$

and the field space Laplacian which is in the kinetic term in the Hamiltonian as

$$- \left(\frac{\delta}{\delta A^a(x)} \right)^2 U_R = C_2(R) U_R \quad (2.17)$$

where $C_2(R)$ is the second Casimir invariant corresponding to the representation R which is obtained from the formula

$$C_2(R) \cdot \mathcal{I} = \sum_a T_R^a T_R^a \quad (2.18)$$

We thus find physical eigenstates of the Hamiltonian by forming traces of U_R , i.e., the characters which are defined by

$$\chi_R(U) = \text{Tr } U_R \quad (2.19)$$

so that

$$H \chi_R(U) = e^2 L C_2(R) \chi_R(U) \quad (2.20)$$

Thus, the action of the Hamiltonian on the characters is proportional to that of the group Laplacian, $\Delta(G)$ whose spectrum is the set of second Casimir invariants of irreducible representations.

Characters are orthonormal with respect to integration over the Haar measure,

$$\int [dU] \chi_R(U) \chi_{R'}^*(U) = \delta_{R,R'} \quad (2.21)$$

This inner product can be obtained from the natural inner product, which is functional integration over A , by gauge fixing,

$$\int dA(x) \Psi_{\text{phys.}}^*[A] \Psi_{\text{phys.}}[A] = \text{const.} \cdot \int [dU] \psi^*(U, \theta) \psi(U, \theta) \quad (2.22)$$

Products of characters are also important as they effectively carry out the multiplication of irreducible representations

$$\chi_{R_1}(U) \chi_{R_2}(U) = \chi_{R_1 \otimes R_2}(U) \quad (2.23)$$

If $R_1 \otimes R_2 = \oplus N_{R_1 R_2}^{R_3} R_3$ is a decomposition into irreducible representations of the product then by linearity of characters

$$\chi_{R_1}(U) \chi_{R_2}(U) = \oplus N_{R_1 R_2}^{R_3} \chi_{R_3}(U) \quad (2.24)$$

Consequently we have the definition of the fusion numbers,

$$N_{R_1 R_2}^{R_3} = \int [dU] \chi_{R_1}(U) \chi_{R_2}(U) \chi_{R_3}^*(U) \quad (2.25)$$

Thus the set of all states is identified with the characters of U . The theta-states in a given theta-sector are a subset of these. For a certain representation of the center of the gauge group, with elements $\mathcal{Z}(z, \theta)$ we must chose the characters for those representations of G which have the property

$$\chi_R(zU) = \mathcal{Z}(z, \theta) \chi_R(U) \quad (2.26)$$

The spectrum of the Hamiltonian is then the set of second Casimir invariants corresponding to the representations R of the group G whose characters have this property.

In the following we shall deal mostly with the special case where the gauge group is $\text{SU}(N)$. Then the $N^2 - 1$ generators T^a are $N \times N$ traceless Hermitean matrices. The center of $\text{SU}(N)$ is the cyclic group of order N , Z_N . The representations which have the property $\chi_R(e^{2\pi i n/N} U) =$

$\mathcal{Z}(\theta)\chi_R(U) = e^{in\theta}\chi_R(U)$ with $\theta = 2\pi k/N$ are those representations whose Young tableaux have k modulo N boxes. If we consider $U(N)$ representations, this is equivalent to keeping only those with linear Casimir $C_1(R) = k \bmod N$.

It is well known that 2D Yang-Mills theory is equivalent to a particular version of unitary matrix quantum mechanics. This can either be seen by fixing a particular gauge as in [10, 11] or by the following argument which shows that the Hilbert space of states, energy levels and degeneracies of the two theories are identical if we identify the unitary matrices as the Wilson loop operator in (2.12). The Hamiltonian of unitary matrix quantum mechanics is proportional to the group Laplacian,

$$H_{QM} = e^2 L \Delta(G) \quad (2.27)$$

can be derived by canonical quantization of the action

$$S = \frac{1}{2e^2 L} \int dt \operatorname{Tr} (\dot{U}^\dagger \dot{U}) = \frac{1}{4e^2 L} \int dt \operatorname{Tr} (iT^a U^\dagger(t) \dot{U}(t))^2 \quad (2.28)$$

The canonical momenta are given by

$$\Pi^a = \frac{1}{2e^2 L} \operatorname{Tr} (iT^a U^\dagger \dot{U}) \quad (2.29)$$

and the Hamiltonian is

$$H_{QM} = e^2 L \sum_a (\Pi^a)^2 = e^2 L \Delta(G) \quad (2.30)$$

The canonical momentum operators have the Lie algebra

$$[\Pi^a, \Pi^b] = if^{abc} \Pi^c \quad (2.31)$$

and

$$[\Pi^a, U] = T^a U, \quad [\Pi^a, U^\dagger] = -U^\dagger T^a, \quad (2.32)$$

The wave-functions of this system are the group elements in unitary irreducible representations of the group,

$$\Delta(G) U_R = C_2(R) U_R \quad (2.33)$$

The inner product is given by the integration over the group with the invariant Haar measure, where the wave-functions have the property

$$\int [dU] (U_R^\dagger)_{kl} (U_R)_{ij} = \frac{1}{\dim R} \delta_{il} \delta_{jk} \quad (2.34)$$

Each component of the unitary matrix in the representation R is a linearly independent, normalizable wave-function. The degeneracy of each eigenstate is equal to the number of linearly independent components, i.e. $(\dim R)^2$.

However, to produce the spectrum and degeneracies of 2D Yang-Mills theory, the wave-functions must be restricted to class functions of group elements so that, for each representation of G , only the wave-function $\chi_R(U)$ is allowed. At the operator level, this restriction is realized by the constraint

$$\sum_a \Pi^a \cdot \operatorname{Tr} (T^a T^b - T^a U T^b U^\dagger) \sim 0, \quad (2.35)$$

The operator in this constraint generates infinitesimal adjoint transformations of the group elements. This constraint can be enforced by a Lagrange multiplier and the Hamiltonian and constraint can be obtained by canonical quantization of the action

$$S = \frac{1}{2e^2 L} \int dt \text{Tr} \left| \dot{U}(t) - i[U(t), A(t)] \right|^2 \quad (2.36)$$

This action has a gauge invariance under

$$A \rightarrow A^g = gAg^\dagger - ig\dot{g}^\dagger, \quad U \rightarrow U^g = gUg^\dagger \quad (2.37)$$

Here, the gauge field $A(t)$ is a Lagrange multiplier enforcing the matrix mechanics analog of Gauss' law. The constraint implies that the adjoint action of the group generators on the wave-function vanishes. Wave-functions which are annihilated by the constraint are class functions of the group variables,

$$\xi[U] = \xi[gUg^\dagger] \quad (2.38)$$

Of the $(\dim(R))^2$ linearly independent eigenfunctions U_R of $\Delta(G)$, only one, the character $\chi_R(U) = \text{Tr} U_R$ has this property.

In this theory, the expectation value of the Wilson loop operator in a given quantum state is given by the fusion number,

$$\langle R_1 | \chi_R(U) | R_2 \rangle = \int [dU] \chi_{R_1}^*(U) \chi_R(U) \chi_{R_2}(U) = N_{R_2 R}^{R_1} \quad (2.39)$$

Similarly, the correlator of a product of Wilson loop operators in arbitrary representations is given by

$$\langle R | \chi_{R_1}(U) \chi_{R_2}(U) \dots \chi_{R_k}(U) | R' \rangle = \int [dU] \chi_R^*(U) \chi_{R_1}(U) \chi_{R_2}(U) \dots \chi_{R_k}(U) \chi_{R'}(U) \quad (2.40)$$

For the group $SU(N)$, explicit formulae for these moments can be obtained. Also, the partition function for this theory at finite temperature $T = \tau^{-1}$ is given the the familiar expression for the partition function of Yang-Mills theory on the torus,

$$Z[\tau] = \sum_R \exp \left(-e^2 L \tau C_2(R) \right) \quad (2.41)$$

We would expect that this partition function could also be written as a functional integral for the d=1 unitary matrix model with action(2.36). We shall show in the following sections that, modulo some subtleties with boundary conditions, this is indeed the case.

3 The heat kernel: Yang-Mills theory on the cylinder

In this Section, we shall give two equivalent presentations of the heat kernel for 2D Yang-Mills theory. The first shows the connection between 2D Yang-Mills theory and a certain principal chiral model which generalizes a result in [17]. In this representation, the dynamical variables are the unitary matrices which would parallel transport heavy quark wave functions from the initial to the final (Euclidean) time. If we consider a partition function, so that Euclidean time is periodic, their traces would be Polyakov loop operators.

Alternatively, we shall find it useful to represent the heat kernel as a $c = 1$ unitary matrix model where the matrices are Wilson loop variables. We shall show, with details in subsection 3.2, the equivalence of the latter model with the well known character expansion of the heat kernel.

We begin by considering the propagation function in Yang-Mills theory,

$$K[\tau; A_2, A_1] \equiv \langle A_2 | e^{-H\tau} P | A_1 \rangle \quad , \quad (3.1)$$

where $|A\rangle$ is an eigenstate of the gauge field operator $A^a(x)$, H is the Hamiltonian in (2.2) and P is a projection operator onto gauge invariant states. For the moment, we consider states which are invariant under periodic gauge transformations only. The projection can be implemented by gauge transforming the field A_1 at one side of the trace and integrating over all gauge transformations,

$$K[\tau; A_2, A_1] = \int [dg(x)] \langle A_2 | e^{-H\tau} | A_1^g \rangle \quad (3.2)$$

(where we normalize the measure so that $\int [dg(x)] = 1$ and $g(0) = g(L)$.) The integrand in (3.2) is the heat kernel which obeys the following equation,

$$\left(\frac{\partial}{\partial \tau} - e^2 \int_0^L dx \sum_a \left(\frac{\delta}{\delta A_2^a(x)} \right)^2 \right) \langle A_2 | e^{-H\tau} | A_1 \rangle = 0 \quad (3.3)$$

with the boundary condition

$$\lim_{\tau \rightarrow 0} \langle A_2 | e^{-H\tau} | A_1^g \rangle = \prod_x \delta(A_2(x) - A_1^g(x)) \quad (3.4)$$

These are solved by⁴

$$K[\tau; A_2, A_1] = \int [dg(x)] \exp \left(-\frac{1}{2e^2\tau} \int_0^L dx \text{Tr}(A_2 - A_1^g)^2 \right) \quad (3.5)$$

We can re-arrange the action in (3.5) to put it in the following form,

$$K[\tau; A_2, A_1] = \int [dg(x)] \exp \left(-\frac{1}{2e^2\tau} \int_0^L dx \text{Tr} |\nabla g + ig(x)A_1(x) - iA_2(x)g(x)|^2 \right) \quad (3.6)$$

This is the path integral for a 0+1-dimensional principal chiral model with external gauge fields where we treat the spatial variable x as Euclidean time. With this identification, it also coincides with the partition function unitary matrix quantum mechanics coupled to external gauge fields. Since the Haar measure has the properties $[dg(x)] = [d(u(x)g(x))] =$

⁴Here, we have dropped a zero point energy term for the Hamiltonian. Also, this equation should be divided by the normalization factor $(2\pi e^2\tau)^{L\delta(0)}$. If we use zeta-function regularization, this normalization has a very simple form. Since

$$L\delta(0) = \lim_{s \rightarrow 0} (1 + 2 \sum_{n=1}^{\infty} 1/n^s) = 0$$

the normalization factor is one.

$[d(g(x)v^\dagger(x))]$ where $u(x)$ and $v(x)$ are unitary matrices, the heat kernel has the property that it is invariant, under gauge transformation of $A_1(x)$ and $A_2(x)$ separately,

$$K[\tau; A_2^u, A_1^v] = K[\tau; A_2, A_1] \quad (3.7)$$

This implies that the heat kernel is a class function of each of the two unitary matrices

$$U_1 = \mathcal{P} \exp \left(i \int_0^L dx A_1(x) \right) , \quad U_2 = \mathcal{P} \exp \left(i \int_0^L dx A_2(x) \right) \quad (3.8)$$

Consequently we define

$$K[\tau; A_2, A_1] = K[\tau; U_2, U_1] \equiv \langle U_2 | e^{-\tau H} P | U_1 \rangle \quad (3.9)$$

which has the invariance property

$$K[\tau; U_2, U_1] = K[\tau; uU_2u^\dagger, vU_1v^\dagger] , \quad (3.10)$$

As is shown in the subsection 3.2, this form of the heat kernel (3.9) has a natural expansion in terms of group characters,

$$K[\tau; U_2, U_1] = \sum_R \chi_R(U_1) \exp \left(-e^2 \tau L C_2(R) \right) \chi_R^*(U_2) \quad (3.11)$$

which, with the expression (3.8) and (2.33) can be seen to satisfy the heat equation

$$\left(\frac{\partial}{\partial \tau} + e^2 L \Delta(G) \right) K[\tau; U_2, U_1] = 0 \quad (3.12)$$

where the gauge group Laplacian $\Delta(G)$ operates on U_2 . Also, the boundary condition,

$$K[0, U_2, U_1] = \int [dg(x)] \prod_x \delta(A_2(x) - A_1^g(x)) = (\text{const.}) \cdot \delta_{\text{cl}}(U_1, U_2) \quad (3.13)$$

The delta function on the right-hand-side of (3.13) is the conjugation invariant delta function which equates the eigenvalues of the two unitary matrices. It can be defined by group integration,

$$\delta_{\text{cl}}(g_1, g_2) = \int [dV] \delta(V g_1 V^\dagger g_2, \mathcal{I}) = \sum_R \chi_R(g_1) \chi_R^*(g_2) \quad (3.14)$$

Furthermore, the heat kernel for 2-dimensional Yang-Mills theory on a cylinder of length τ and base circle L can be written in terms of the partition function of a gauged principal chiral model with open boundaries. This, as we showed in Section 2, is equivalent to 2D QCD if we restrict our attention to Wilson loops. The path integral representation of the heat kernel for that theory should have a standard form using action (2.36) and integration variables A and U . Some care must be taken to ensure that the heat kernel is a correct function of the eigenvalues of the Wilson loop operators. In particular the sewing property,

$$Z[\tau; U_2, U_1] = \int dU(u) Z[u; U_2, U(u)] Z[\tau - u; U(u), U_1] , \quad (3.15)$$

must be satisfied.

The conditions that fix the loops at the two boundaries of the cylinder (say $U(0)$ and $U(\tau)$) to be U_1 and, up to an element of Z_N , U_2 , can be imposed introducing the corresponding delta-functions:

$$Z[\tau; U_1, U_2] = \int \prod_{t \in [0, \tau]} [dA(t)][dU(t)] e^{-\frac{1}{2e^2 L} \int_0^\tau dt \text{Tr} |\dot{U} - i[A, U]|^2} \delta_{\text{cl}}(U(0), U_1) \delta_{\text{cl}}(U(\tau), U_2) \psi(U(0)) \psi(U(\tau)) . \quad (3.16)$$

The factors $\psi(U(0))$ and $\psi(U(\tau))$ are boundary wave functions that have to be introduced in order to guarantee that the sewing prescription (3.15) for the kernel is satisfied. By means of eq.(3.16), the left hand side of eq.(3.15) reads

$$\int dU(u) \prod_{t \in [0, u]} [dA(t)][dU(t)] \prod_{t \in [u, \tau]} [dA'(t)][dU'(t)] e^{-\frac{1}{2e^2 L} \int_0^u dt \text{Tr} |\dot{U} - i[A, U]|^2} e^{-\frac{1}{2e^2 L} \int_u^\tau dt \text{Tr} |\dot{U}' - i[A', U']|^2} \delta_{\text{cl}}(U(0), U_1) \delta_{\text{cl}}(U(u), U(u)) \delta_{\text{cl}}(U'(u), U(u)) \delta_{\text{cl}}(U'(\tau), U_2) \psi(U(0)) \psi(U(u)) \psi(U'(u)) \psi(U'(\tau)) . \quad (3.17)$$

We will show in the next section that the integration over one of the A variables at the point u , produces the squared inverse of the Vandermonde determinant for a unitary matrix, $J^{-2}(U(u))$

$$J(U) = \prod_{\alpha < \beta} 2 \sin \frac{1}{2} (\phi^\alpha - \phi^\beta) \quad (3.18)$$

which only depends on the eigenvalues of U , $\exp(i\phi^\alpha)$, $\alpha = 1, \dots, N$. Consequently, Eq.(3.15) will be satisfied if we choose $\psi(U) = J(U)$, so that

$$K[\tau; U_2, U_1] = \int [dA(t)][dU(t)] \exp \left(-\frac{1}{2e^2 L} \int_0^\tau dt \text{Tr} |\dot{U}(t) - i[A(t), U(t)]|^2 \right) \cdot \delta_{\text{cl}}(U(0), U_1) \delta_{\text{cl}}(U(\tau), U_2) J(U(0)) J(U(\tau)) . \quad (3.19)$$

The formulae (3.6) and (3.19) are two alternative expressions for the heat kernel, one in terms of matrix quantum mechanics with a background gauge field and the other in terms of gauge invariant matrix mechanics with the external matrices appearing as boundary conditions.

The propagator in a θ -sector can be obtained by projecting either the initial or final state of the heat kernel onto θ -states. This is done by summing over all transformations of U_1 or U_2 by elements of the center of G , C . and, in that sum, weighting each term by the phases $\mathcal{Z}(z, \theta)$ corresponding to the θ -representation of C ,

$$K[\tau, \theta; U_2, U_1] = \sum_{z \in C} \mathcal{Z}(z, \theta) K[\tau; zU_2, U_1] \quad (3.20)$$

For the three presentations of the heat kernel given above, this has the effect

$$\begin{aligned} K[\tau, \theta; U_2, U_1] &= \sum_R \delta(R, R_\theta) \chi_R^*(U_2) \chi_R(U_1) \cdot \exp \left(-e^2 L \tau C_2(R) \right) \\ &= \sum_z \mathcal{Z}(z, \theta) \int [dg_z(x)] \exp \left(-\frac{1}{2e^2 \tau} \int_0^L dx \text{Tr} |\nabla g_z(x) + ig_z(x) A_1(x) - i A_2(x) g_z(x)|^2 \right) \end{aligned} \quad (3.21)$$

$$\begin{aligned} &= \sum_z \mathcal{Z}(z, \theta) \int [dA(t)][dU(t)] \exp \left(-\frac{1}{2e^2 L} \int_0^\tau dt \text{Tr} |\dot{U}(t) - i[A(t), U(t)]|^2 \right) \cdot \\ &\quad \cdot \delta_{\text{cl}}(U(0), U_1) \delta_{\text{cl}}(U(\tau), zU_2) J(U(0)) J(U(\tau)) \end{aligned} \quad (3.23)$$

Here, R_θ are those irreducible representations of G which have the property

$$\chi_{R_\theta}(zU) = \mathcal{Z}(z, \theta)\chi_{R_\theta}(U) .$$

In the next section, by means of a diagonalization technique, we shall show that the functional integral in (3.23) can be actually performed so that the heat kernel can be expressed directly in terms of the eigenvalues of U_1 and U_2 .

3.1 Calculation of the heat kernel by diagonalization technique

In Ref.[12] two dimensional Yang-Mills theories were written in terms of a Kazakov-Migdal model and the heat kernel on the cylinder was computed using a diagonalization procedure. We shall follow here an analogous procedure for the calculation of the partition function (3.23) of the gauged principal chiral model. In fact, the integral in (3.23) can be done by using gauge symmetry to diagonalize at each point the unitary matrix $U(t) = V(t)U^D(t)V^{-1}(t)$ where $U^D(t) = \text{diag}(e^{i\varphi^1(t)}, e^{i\varphi^2(t)}, \dots, e^{i\varphi^N(t)})$. The φ variables are angles since $\varphi \in [0, 2\pi]$. The measure in the integral has the form

$$[dU(t)] = \prod_{t \in [0, \tau]} \prod_{\alpha} d\varphi^\alpha(t) J^2(\varphi(t)) \delta \left(\sum_{\alpha} \varphi^\alpha(t) \right) [dV(t)] , \quad (3.24)$$

where $J(\varphi(t))$ is the Vandermonde determinant for a unitary matrix, and the action is

$$S = \frac{1}{2e^2 L} \int_0^\tau dt \left(\sum_{\alpha=1}^N \dot{\varphi}^\alpha \dot{\varphi}^\alpha + \sum_{\alpha, \beta=1}^N |A_{\alpha\beta}|^2 |e^{i\varphi^\alpha} - e^{i\varphi^\beta}|^2 \right) . \quad (3.25)$$

The integral over $A_{\alpha\beta}$, where $\alpha \neq \beta$, cancels the Vandermonde determinant in the integration measure. The integration over the diagonal components of A yields an infinite factor which compensates the infinite normalization of the plane-wave states which were used in (3.9). The kernel (3.23) becomes an integral over the eigenvalue variables. In the θ -sector using for $\sum_z \mathcal{Z}(z, \theta)$ the representation $\sum_n \exp(-in\theta_k)$ with $\theta_k = 2\pi k/N$, we get

$$\begin{aligned} Z = & \sum_{n=0}^{N-1} e^{-i\theta_k n} \int \prod_{t \in [0, \tau]} \prod_{\alpha=1}^N d\varphi^\alpha(t) \delta \left(\sum_{\alpha=1}^N \varphi^\alpha(t) \right) e^{-\frac{1}{2e^2 L} \int_0^\tau dt \sum_{\alpha=1}^N \dot{\varphi}^\alpha \dot{\varphi}^\alpha} J(\varphi(0)) J(\varphi(\tau)) \\ & \int dV(0) \delta \left(V(0) e^{i\varphi(0)} V^\dagger(0) e^{-i\lambda_1}, I \right) \int dV(\tau) \delta \left(V(\tau) e^{i\varphi(\tau)} V^\dagger(\tau) e^{-i\lambda_2 + 2\pi i n/N}, I \right) \end{aligned} \quad (3.26)$$

where $e^{i\lambda_{1,2}}$ denote the diagonal forms of $U_{1,2}$ and the integration measure integrates $\varphi(t)$, at each point t , over the range $[0, 2\pi]$. The integral is invariant under the field translation symmetry

$$\varphi^\alpha(t) \rightarrow \varphi^\alpha(t) + 2\pi n^\alpha \quad (3.27)$$

where n^α is an integer.⁵ This symmetry can be used to extend the limits on the integration over $\varphi(t)$ at each t to infinity.

⁵Actually in this case n_α can be any constant, but when computing loop correlators only the symmetry (3.27) with integer n_α will survive. Note that this is not a symmetry of the action or the integration measure in the path integral (3.23) separately but only appears after diagonalization.

The integrals on $\varphi^\alpha(t)$ with $t \in (0, \tau)$ can be performed by a ζ -function regularization of the divergences. First one can integrate on $\varphi^N(t)$ taking advantage of the $\delta(\sum_\alpha \varphi^\alpha(t))$. Changing the functional integration variables to the real and periodic $\tilde{\varphi}^\alpha(t)$, $\tilde{\varphi}^\alpha(\tau) = \tilde{\varphi}^\alpha(0)$, by means of

$$\varphi^\alpha(t) = (\varphi^\alpha(\tau) - \varphi^\alpha(0)) \frac{t}{\tau} + \tilde{\varphi}^\alpha(t) \quad , \quad (3.28)$$

the integrals on the open interval $t \in (0, \tau)$, give

$$\exp \left\{ -\frac{1}{2e^2 L\tau} \left[\sum_{\alpha, \beta=1}^{N-1} (\varphi^\alpha(\tau) - \varphi^\alpha(0)) \Omega_{\alpha\beta} (\varphi^\beta(\tau) - \varphi^\beta(0)) \right] \right\} \cdot Z_a \quad , \quad (3.29)$$

where

$$Z_a = \frac{1}{\text{VOL } G} \int \prod_{t \in (0, \tau)} \prod_{\alpha=1}^{N-1} d\tilde{\varphi}^\alpha(t) \exp \left(-\frac{1}{2e^2 L\tau} \int_0^\tau dt \sum_{\alpha\beta=1}^{N-1} \dot{\tilde{\varphi}}^\alpha \Omega_{\alpha\beta} \dot{\tilde{\varphi}}^\beta \right) \quad (3.30)$$

and Ω is the $(N-1) \times (N-1)$ matrix

$$\Omega = \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{pmatrix} \quad . \quad (3.31)$$

Expanding $\tilde{\varphi}^\alpha(t)$ in modes

$$\varphi^\alpha(t) = \frac{1}{\sqrt{\tau}} \sum_{k=-\infty}^{\infty} a_k^\alpha e^{i2\pi kt/\tau} \quad (3.32)$$

the functional measure is defined as $\prod_t d\varphi(t) \equiv \prod_k da_k$. The integration over the zero modes produce an infinite, irrelevant, temperature independent factor proportional to the volume of the gauge group. The functional integral in Z_a is proportional to the determinant of the Laplacian,

$$Z_a = \prod_{k \neq 0} \left(\det \frac{e^2 L\tau}{2\pi} \Omega k^2 \right)^{-\frac{1}{2}} = \det \left(\frac{e^2 L\tau}{2\pi} \Omega \right)^{-\zeta(0)} e^{(N-1)\zeta'(0)} = \sqrt{N} \left(\frac{1}{e^2 L\tau} \right)^{\frac{N-1}{2}} \quad . \quad (3.33)$$

where we have used zeta-function regularization with

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^{2s}} \quad (3.34)$$

and $\zeta(0) = -1/2$ and $\zeta'(0) = -(\log 2\pi)/2$.

We are now left with the integration on the $\varphi^\alpha(t)$, $\alpha = 1, \dots, N$ at the boundaries $t = 0, \tau$. This can be easily done by rewriting the conjugate invariant delta functions in (3.26) according to [12, 39]

$$\int dV \delta(V e^{i\varphi} V^\dagger e^{-i\lambda}) = \sum_P \sum_{\{n_\alpha\}=-\infty}^{+\infty} \frac{(-1)^{P+(N-1)\sum_{\alpha=1}^N n_\alpha}}{J(\varphi)J(\lambda)} \prod_{\alpha=1}^N \delta(\varphi_\alpha - \lambda_{P(\alpha)} + 2\pi n_\alpha) \quad , \quad (3.35)$$

where P denotes a permutation of the indices and $(-1)^P$ its parity. By substituting Eqs.(3.29), (3.33), (3.35) into Eq.(3.26) and integrating in $\varphi^\alpha(0)$, $\varphi^\alpha(\tau)$ we finally get

$$K[\tau, \theta_k; U_2, U_1] = N! \sqrt{N} \left(\frac{1}{e^2 L \tau} \right)^{\frac{N-1}{2}} \sum_{n=0}^{N-1} \sum_P \frac{(-1)^P e^{-2\pi i n [k - N(N-1)/2]/N}}{J(\lambda_1) J(\lambda_2)} \\ \sum_{\{n_\alpha\}=-\infty}^{+\infty} \delta \left(\sum_{\alpha=1}^N n_\alpha + n \right) \exp \left[-\frac{1}{2e^2 L \tau} \sum_{\alpha=1}^N \left(\lambda_1^\alpha - \lambda_2^{P(\alpha)} + 2\pi n_\alpha + \frac{2\pi n}{N} \right)^2 \right] . \quad (3.36)$$

The delta-function imposing the constraint $\sum_\alpha n_\alpha + n = 0$ arises from the $\delta(\sum_\alpha \varphi_\alpha)$ by taking into account that the U_2 eigenvalues are twisted by an element of the center and that, being the gauge group $SU(N)$, we can choose $\sum_\alpha \lambda_\alpha = 0$ for the loops at the boundaries. We shall show in the next subsection that Eq.(3.36) is equivalent to the character representation (3.21).

3.2 From the functional integral to the character expansion

Following Ref.[12], it is possible to show that, by means of a Poisson resummation, Eq.(3.36) can be rewritten in terms of a character expansion. It is convenient first to rewrite the delta function on the integers n_α as an integral

$$\delta \left(\sum_{\alpha=1}^N n_\alpha + n \right) = \int_0^{2\pi} \frac{d\vartheta}{2\pi} \exp \left(i\vartheta \sum_{\alpha=1}^N n_\alpha + n \right) . \quad (3.37)$$

One can then complete the square in n_α , so as to rewrite Eq.(3.36), according to

$$K[\tau, \theta_k; U_1, U_2] = N! \left(\frac{1}{e^2 L \tau} \right)^{\frac{N-1}{2}} \sum_{n=0}^{N-1} \sum_P \frac{(-1)^P e^{-2\pi i n [k - N(N-1)/2]/N}}{J(\lambda_1) J(\lambda_2)} \\ \sum_{\{n_\alpha\}=-\infty}^{+\infty} \int_0^{2\pi} \frac{d\vartheta}{2\pi} \exp \left[-\frac{2\pi^2}{e^2 L \tau} \sum_{\alpha=1}^N \left(\frac{\lambda_1^\alpha - \lambda_2^{P(\alpha)}}{2\pi} + n_\alpha + \frac{n}{N} - \frac{i\vartheta L \tau e^2}{4\pi^2} \right)^2 - \frac{N\vartheta^2 e^2 L \tau}{8\pi^2} \right] , \quad (3.38)$$

where we used that $\sum_{\alpha=1}^N \lambda_\alpha^i = 0$, $i = 1, 2$. Applying the generalized Poisson resummation formula

$$\sum_{\vec{m}=-\infty}^{+\infty} \exp \left(-\pi g_{ij} m^i m^j - 2\pi i m^i a_i \right) = \frac{1}{\sqrt{\det g_{ij}}} \sum_{\vec{n}=-\infty}^{+\infty} \exp \left[-\pi g^{ij} (n_i - a_i)(n_j - a_j) \right] . \quad (3.39)$$

where g^{ij} is g_{ij} inverse, and then completing the square in ϑ , Eq.(3.38) becomes

$$K[\tau, \theta_k; U_1, U_2] = \left(e^2 L \tau N \right)^{1/2} \sum_{\{m_\alpha\}} \sum_{n=0}^{N-1} \sum_{P, P'} \frac{(-1)^{(P+P')} e^{-2\pi i n [k - N(N-1)/2]/N}}{J(\lambda_1) J(\lambda_2)} \\ \exp \left[i \sum_{\alpha=1}^N m_\alpha \left(\lambda_1^{P(\alpha)} - \lambda_2^{P'(\alpha)} + 2\pi i n \frac{1}{N} \right) \right] \exp \left\{ -\frac{e^2 L \tau}{2} \left[\sum_{\alpha=1}^N m_\alpha^2 - \frac{1}{N} \left(\sum_{\alpha=1}^N m_\alpha \right)^2 \right] \right\} \\ \left(\frac{1}{2\pi} \right)^{(1+N/2)} \int_0^{2\pi} d\vartheta \exp \left[-\frac{N e^2 L \tau}{8\pi^2} \left(\vartheta - \frac{2\pi}{N} \sum_{\alpha=1}^N m_\alpha \right)^2 \right] , \quad (3.40)$$

where we introduced a redundant sum over permutation canceling the $N!$ factor. Note that Eq.(3.40) is symmetric with respect to any permutation of the m_α and vanishes when any two of m_α , m_β are equal. Therefore one can introduce an ordering between the integers $m_1 > m_2 > \dots > m_N$ multiplying by a factor $N!$. One can further notice that all the terms, except the last integral are invariant if all the m_α are shifted by the same integer. One can then define the integers $r_\alpha = m_\alpha - m_N - N$, $\alpha = 1, \dots, N$ and use the sum over m_N to complete the Gaussian integral in ϑ by shifting $\vartheta \rightarrow \vartheta - 2\pi m_N - 2\pi N$. These manipulations lead to

$$K[\tau, \theta_k; U_1, U_2] = N! \sum_{n=0}^{N-1} \sum_{r_1 > r_2 > \dots > r_N = -N} \exp \left\{ -\frac{2\pi i n}{N} \left[k - \frac{N(N-1)}{2} - \sum_{\alpha=1}^N r_\alpha \right] \right\} \\ \left(\frac{1}{2\pi} \right)^{(N-1)/2} \frac{\det e^{ir_\alpha \lambda_\beta^1}}{J(\lambda_1)} \frac{\det e^{-ir_\alpha \lambda_\beta^2}}{J(\lambda_2)} \exp \left\{ -\frac{e^2 L \tau}{2} \left[\sum_{\alpha=1}^N r_\alpha^2 - \frac{1}{N} \left(\sum_{\alpha=1}^N r_\alpha \right)^2 \right] \right\} \quad (3.41)$$

The ingredients of Eq.(3.41) are now easy to recognize. The last exponent is, up to a constant, the $SU(N)$ quadratic Casimir in the representation R whose Young table has $r_\alpha + \alpha$ boxes in the α^{th} row.

$$\sum_{\alpha=1}^N r_\alpha^2 - \frac{1}{N} \left(\sum_{\alpha=1}^N r_\alpha \right)^2 = 2C_2(R) + \frac{N(N^2 - 1)}{12}. \quad (3.42)$$

The second term in the r.h.s. of Eq.(3.42) is the trace of the inverse Cartan matrix and is related to the scalar curvature of the group manifold. It provides the energy of the lowest representation. The first exponential in Eq.(3.41) projects the sum to those representations R_k whose Young tableaux have rows with a number of boxes equal to the theta vacuum k , mod $[N]$. Moreover the determinants are related to the character of the matrix $U_{1,2}$ in the representation R .

$$\chi_R(U) = \frac{\det(e^{ir_\alpha \lambda_\beta})}{J(\lambda)}. \quad (3.43)$$

Up to an irrelevant, temperature independent constant, one can then write the final equation as

$$K[\tau, \theta_k; U_1, U_2] = \sum_{R_k} \chi_{R_k}(U_1) \chi_{R_k}^*(U_2) \exp \left\{ -\frac{e^2 L \tau}{2} \left[2C_2(R_k) + \frac{N(N^2 - 1)}{12} \right] \right\}. \quad (3.44)$$

3.3 2D Yang-Mills theory on the sphere as matrix quantum mechanics

Once one has the heat kernel for 2D Yang-Mills it is an easy matter to construct the partition function for the theory on a sphere or a torus. If one imagines shrinking the ends of the cylinder to points then the resulting topology is that of a sphere. In terms of the the discussion of the previous section this corresponds to setting $U_1 = U_2 = 1$. Consequently, from (3.11) the partition function on the sphere is

$$Z[S^2; \tau, \theta] = K[\tau; \theta, 1, 1] = \sum_R \delta(R, R_\theta) (\dim(R))^2 \exp(-e^2 \tau L C_2(R)) \quad (3.45)$$

where we have used the fact that $\chi_R(1) = \dim(R)$ and R_θ is defined in (4).

More interestingly, if we carry out the equivalent procedure with the path integral principal chiral model of the heat kernel (3.6) and set $A_1(x) = A_2(x) = 0$ we find the following form

$$Z[S^2; \tau, \theta] = \sum_z \mathcal{Z}(z, \theta) \int [dg_z(x)] \exp \left(-\frac{1}{2e^2\tau} \int_0^L dx \text{Tr} |\nabla g_z(x)|^2 \right) \quad (3.46)$$

where $g(L) = zg(0)$.

Consequently we can interpret the partition function for the sphere as unitary matrix quantum mechanics.

3.4 2D Yang-Mills theory on the torus as gauge invariant matrix quantum mechanics

Similar to the case of the sphere it is easy to imagine manipulating a cylinder to form a torus - identifying the ends and sewing them together - which produces the trace of the heat kernel. This process is easy to carry out in the character representation by identifying $U_1 = U_2$ and integrating. Hence we have the partition function on a torus of area τL

$$Z[T^2; \tau, \theta] = \int [dU] K[\tau, \theta; U, U] = \sum_R \delta(R, R_\theta) \exp \left(-e^2 \tau L C_2(R) \right) \quad (3.47)$$

Explicitly, for the special case of $SU(N)$, where representations are denoted by the usual Young Tableau row variables $l_1 \geq l_2 \geq \dots \geq l_N \geq 0$ with $l = \sum l_j$ we can give an explicit form for the partition function on the torus. When $\theta = 2\pi k/N$, we consider only those representations where $\sum_i l_i = k \pmod{N}$. Then

$$Z[\tau, \theta_k = 2\pi k/N] = \sum_{\{l_j\}} \delta_N(\sum_i l_i, k) \exp \left[-e^2 \tau L \sum_{j=1}^N l_j (l_j + N + 1 - 2j - l/N) \right] \quad (3.48)$$

Equivalently, in the following section, we shall show how this result can be obtained by taking the trace of the heat kernel represented as a functional integral for the matrix model (3.36).

3.5 Partition function on the torus in the functional integral approach

From the kernel on the cylinder obtained in the section 3.1, Eq.(3.36), the partition function on the torus can be readily obtained by sewing together the two ends of the cylinder. Namely one takes the trace of the kernel on the cylinder with the appropriate measure $\prod_\alpha d\lambda_\alpha \delta(\sum_\alpha \lambda_\alpha) J^2(\lambda)$

$$Z[\tau, \theta_k] = \int dU K[\tau, \theta_k; U, U] = N! \sqrt{N} \left(\frac{1}{e^2 L \tau} \right)^{\frac{N-1}{2}} \sum_{n=0}^{N-1} \sum_P (-1)^P \exp \left\{ -2\pi i n \frac{1}{N} \left[k - \frac{N(N-1)}{2} \right] \right\} \sum_{\{n_\alpha\}=-\infty}^{+\infty} \delta \left(\sum_{\alpha=1}^N n_\alpha + n \right) \int_0^{2\pi} \prod_{\alpha=1}^N d\lambda_\alpha \delta \left(\sum_{\alpha=1}^N \lambda_\alpha \right)$$

$$\exp \left[-\frac{1}{2e^2 L\tau} \sum_{\alpha=1}^N \left(\lambda_\alpha^1 - \lambda_{P(\alpha)}^2 + 2\pi n_\alpha + \frac{2\pi n}{N} \right)^2 \right] . \quad (3.49)$$

To calculate the integral in (3.49) we can proceed as in the previous section, first writing the $\delta(\sum_\alpha n_\alpha + n)$ as an integral, completing the square in n_α and then Poisson resumming. One gets

$$\begin{aligned} Z[\tau, \theta_k] = & N! \left(\frac{1}{2\pi} \right)^{1+\frac{N}{2}} (Ne^2 L\tau)^{\frac{1}{2}} \sum_{n=0}^{N-1} \sum_P (-1)^P \sum_{\{n_\alpha\}=-\infty}^{+\infty} e^{-2\pi i n \frac{1}{N} [k - \frac{N(N-1)}{2} - \sum_\alpha n_\alpha]} \\ & \int_0^{2\pi} \prod_{\alpha=1}^N d\lambda_\alpha \delta \left(\sum_{\alpha=1}^N \lambda_\alpha \right) \exp \left\{ -\frac{e^2 L\tau}{2} \left[\sum_{\alpha=1}^N n_\alpha^2 - \frac{1}{N} \left(\sum_{\alpha=1}^N n_\alpha \right)^2 \right] + i \sum_{\alpha=1}^N (n_\alpha - n_{P(\alpha)}) \lambda_\alpha \right\} \\ & \int_0^{2\pi} d\vartheta \exp \left[-\frac{Ne^2 L\tau}{8\pi^2} \left(\vartheta - \frac{2\pi}{N} \sum_{\alpha=1}^N n_\alpha \right)^2 \right] . \end{aligned} \quad (3.50)$$

We can now shift the integer $n_\alpha \rightarrow n_\alpha - n_N - N = r_\alpha$ $\alpha = 1, \dots, N$ and use the sum over n_N to complete the Gaussian integral in ϑ . The $\delta(\sum_\alpha \lambda_\alpha)$ can be eliminated integrating in λ_N . The integrals in λ_α , $\alpha = 1, \dots, N-1$ give just a product of Kronecker deltas imposing the $N-1$ conditions

$$r_\alpha - r_N - r_{P(\alpha)} + r_{P(N)} = 0 , \quad \alpha = 1, \dots, N-1 , \quad (3.51)$$

these can actually be extended to $\alpha = N$, because the N^{th} is trivially satisfied. The sum of the conditions (3.51), for $\alpha = 1, \dots, N$, gives $r_N - r_{P(N)} = 0$, so that the product of Kronecker give rise to the determinant factor

$$\sum_P (-1)^P \prod_{\alpha=1}^N \delta_{r_{P(\alpha)}, r_\alpha} = \det_{\alpha\beta} \delta_{r_\alpha r_\beta} . \quad (3.52)$$

The $SU(N)$ partition function on the torus then reads

$$\begin{aligned} Z[\tau, \theta_k] = & N! (2\pi)^{(N-1)/2} \sum_{n=0}^{N-1} \sum_{\{r_1, \dots, r_{N-1}\}=-\infty}^{+\infty} \exp \left\{ -2\pi i n \frac{1}{N} \left[k - \frac{N(N-1)}{2} - \sum_{\alpha=1}^N r_\alpha \right] \right\} \\ & \det_{\alpha\beta} [\delta_{r_\alpha, r_\beta}] \exp \left\{ -\frac{e^2 L\tau}{2} \left[\sum_{\alpha=1}^N r_\alpha^2 - \frac{1}{N} \left(\sum_{\alpha=1}^N r_\alpha \right)^2 \right] \right\} . \end{aligned} \quad (3.53)$$

The determinant of Kronecker deltas forbids any two of the integers r_α $\alpha = 1, \dots, N$ from taking the same value and, the expression (3.53) being symmetric under the permutation of any r_α , the integers can be ordered according to $r_1 > \dots > r_N = -N$ provided we multiply by a factor $N!$.

The result (3.53) gives the correct spectrum for 2-dimensional $SU(N)$ Yang-Mills theories on the torus which, without the theta angle, has already been provided in many papers [7, 1, 3]; it does not agree with the one found in ref.[18]. $Z_{SU(N)}$ is a sum over the representation of the exponential of the second Casimir, sum which is actually restricted only to those representations whose Young tableaux have rows with $k \bmod [N]$ boxes, where k is the discrete theta angle of 2-dimensional Yang-Mills theories. To compute (3.53) we followed the path integral calculation in [12] generalizing it to the case of non trivial theta states.

4 Loop Correlators on the Cylinder

In this section we shall compute, on the cylinder, the correlator of two Wilson loops in the fundamental representation. One loop is situated at the point u with l windings and the other, at the point v , with m .

$$P_{l,m}(\theta_k, U_1, U_2; u, v) = \frac{1}{N^2 Z_k} \left\langle U_1 \left| \text{Tr}\{U^l(u)\} \text{Tr}\{U^m(v)\} P_k \right| U_2 \right\rangle, \quad (4.1)$$

where Z_k coincides with the kernel on the cylinder in the k^{th} θ -sector (*e.g.* Eq.(3.36)). This will allow us to obtain also the correlator on the torus just by sewing together the two ends of the cylinder, as we did for the partition function in the previous section. Using the prescriptions introduced for the partition function on the cylinder, the path integral representation of the correlator (4.1) reads

$$P_{l,m}(\theta_k, U_1, U_2; u, v) = \frac{1}{N^2 Z_k} \sum_n e^{-in\theta_k} \int \prod_{t \in [0, \tau]} [dA(t)] [dU(t)] e^{-\frac{1}{2e^2 L} \int_0^\tau dt \text{Tr} |\dot{U} - i[A, U]|^2} \\ \text{Tr}\{U^l(u)\} \text{Tr}\{U^m(v)\} \delta_{\text{cl}}(U(0), U_1) \delta_{\text{cl}}(U(\tau), e^{2\pi i n/N} U_2) \psi(U(0)) \psi(U(\tau)). \quad (4.2)$$

Taking, for the time being, $v > u$, in order to compute (4.2) it is most convenient to perform first the integration in the open intervals $t \in (0, u)$, (u, v) and (v, τ) . This corresponds to the calculation of the partition function on the three cylinders with boundaries $(U_1, U(u))$, $(U(u), U(v))$ and $(U(v), U_2)$. After the diagonalization and the integration on the gauge potential in the whole interval $t \in [0, \tau]$, we can perform the path integral in the open intervals $t \in (0, u)$, (u, v) and (v, τ) as we did in the previous section in the open interval $t \in (0, \tau)$. The results will then be given by Eqs.(3.29) and (3.33) with the appropriate changes for the different lengths of the three cylinders. Then, using Eq.(3.35) for the delta-functions in (4.2) one can integrate on the boundary points $t = 0$, $t = \tau$. The result reads

$$P_{l,m}(\theta_k, \lambda_1, \lambda_2; u, v) = \frac{1}{\sqrt{N} Z_k} \left(\frac{1}{(e^2 L)^3 u(v-u)(\tau-v)} \right)^{\frac{N-1}{2}} \sum_{n=0}^{N-1} \sum_{P, P'} \frac{(-1)^{P+P'} e^{-2\pi i n[k-N(N-1)/2]/N}}{J(\lambda_1) J(\lambda_2)} \\ \sum_{\{n_\alpha, l_\alpha\}=-\infty}^{+\infty} \delta \left(\sum_{\alpha=1}^N n_\alpha \right) \delta \left(\sum_{\alpha=1}^N l_\alpha + n \right) \int \prod_{\alpha=1}^N d\varphi^\alpha(u) d\varphi^\alpha(v) \sum_{\beta, \gamma=1}^N e^{i l \varphi^\beta(u) + i m \varphi^\gamma(v)} \\ \delta \left(\sum_{\alpha=1}^N \varphi^\alpha(u) \right) \delta \left(\sum_{\alpha=1}^N \varphi^\alpha(v) \right) \exp \left[-\frac{1}{2e^2 L u} \sum_{\alpha=1}^N \left(\lambda_1^\alpha - \varphi^{P(\alpha)}(u) + 2\pi n_\alpha \right)^2 \right] \\ \exp \left[-\frac{1}{2e^2 L (v-u)} \sum_{\alpha=1}^N \left(\varphi^\alpha(u) - \varphi^\alpha(v) \right)^2 \right. \\ \left. - \frac{1}{2e^2 L (\tau-v)} \sum_{\alpha=1}^N \left(\varphi^\alpha(v) - \lambda_2^{P'(\alpha)} + 2\pi l_\alpha + 2\pi \frac{n}{N} \right)^2 \right], \quad (4.3)$$

where we have used the delta-functions in (4.3) to eliminate the matrix Ω appearing in (3.29). We can now proceed as before: write the delta-functions in the integers as in Eq.(3.37), perform a generalized Poisson resummation in n_α and l_α to the new integers n'_α and l'_α , introduce the integers $r_\alpha = n'_\alpha - n'_N - N$ and $s_\alpha = l'_\alpha - l'_N - N$ ($r_N = s_N = -N$) and integrate

on the variables introduced to impose the constraints on the integers (say ϑ and ϑ') using the sums over n'_N and l'_N to complete the Gaussian integrals in ϑ and ϑ' .

$$\begin{aligned}
P_{l,m}(\theta_k, \lambda_1, \lambda_2; u, v) = & \quad (4.4) \\
& \frac{1}{N^{3/2} Z_k} \left(\frac{1}{4e^2 L \pi^2 (v-u)} \right)^{\frac{N-1}{2}} \sum_{n=0}^{N-1} \sum_{\{r_\alpha, s_\alpha\}=-\infty}^{N-1} \sum_{P, P'} (-1)^{P+P'} \frac{e^{-2\pi i n [k - N(N-1)/2 - \sum_\alpha s_\alpha]/N}}{J(\lambda_1) J(\lambda_2)} \\
& \exp \left\{ -\frac{e^2 L u}{2} \left[\sum_{\alpha=1}^N r_\alpha^2 - \frac{1}{N} \left(\sum_{\alpha=1}^N r_\alpha \right)^2 \right] - \frac{e^2 L (\tau - v)}{2} \left[\sum_{\alpha=1}^N s_\alpha^2 - \frac{1}{N} \left(\sum_{\alpha=1}^N s_\alpha \right)^2 \right] \right\} \\
& \exp \left[i \sum_{\alpha=0}^N \left(r_\alpha \lambda_1^{P(\alpha)} - s_\alpha \lambda_2^{P'(\alpha)} \right) \right] \int \prod_{\alpha=1}^N d\varphi^\alpha(u) d\varphi^\alpha(v) \sum_{\beta, \gamma=1}^N \delta \left(\sum_{\alpha=1}^N \varphi^\alpha(u) \right) \delta \left(\sum_{\alpha=1}^N \varphi^\alpha(v) \right) \\
& \exp \left[i l \varphi^\beta(u) + i m \varphi^\gamma(v) - i \sum_{\alpha=0}^N (r_\alpha \varphi^\alpha(u) - s_\alpha \varphi^\alpha(v)) - \frac{1}{2e^2 L (v-u)} \sum_{\alpha=1}^N (\varphi^\alpha(u) - \varphi^\alpha(v))^2 \right]
\end{aligned}$$

Taking advantage of the delta-functions in Eq.(4.4) to integrate on $\varphi^N(u)$ and $\varphi^N(v)$ the integral in (4.4) becomes

$$\begin{aligned}
& \int \prod_{\alpha=1}^{N-1} d\varphi^\alpha(u) d\varphi^\alpha(v) \sum_{\beta, \gamma=1}^N \exp \left[-i \sum_{\alpha=0}^{N-1} (r_\alpha - r_N - l(\delta_{\alpha\beta} - \delta_{N\beta})) \varphi^\alpha(u) \right] \\
& \exp \left[i \sum_{\alpha=0}^{N-1} (s_\alpha - s_N - m(\delta_{\alpha\gamma} - \delta_{N\gamma})) \varphi^\alpha(v) \right] \\
& \exp \left[-\frac{1}{2e^2 L (v-u)} \sum_{\alpha\delta=1}^{N-1} (\varphi^\alpha(u) - \varphi^\alpha(v)) \Omega_{\alpha\delta} (\varphi^\delta(u) - \varphi^\delta(v)) \right] . \quad (4.5)
\end{aligned}$$

We can now complete the square, for example in $\varphi^\alpha(u)$, using the inverse of the matrix Ω ,

$$\Omega_{\alpha\beta}^{-1} = \frac{1}{N} \begin{pmatrix} N-1 & -1 & \dots & -1 \\ -1 & N-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & N-1 \end{pmatrix} , \quad (4.6)$$

and then perform the Gaussian integration. At this point the integrals on the $\varphi^\alpha(v)$ just give products of Kronecker deltas relating the integers s_α to the r_α through the conditions

$$s_\alpha = r_\alpha - n(\delta_{\alpha\beta} - \delta_{N\beta}) - m(\delta_{\alpha\gamma} - \delta_{N\gamma}) . \quad (4.7)$$

Summing on the s_α , we finally get, for the $SU(N)$ correlator of two loops on a cylinder ($v > u$)

$$\begin{aligned}
P_{l,m}(\theta_k, \lambda_1, \lambda_2; u, v) = & \frac{1}{N^2 Z_k} \left(\frac{1}{2\pi} \right)^{\frac{N-1}{2}} \sum_{\{r_1, \dots, r_{N-1}\}=-\infty}^{+\infty} \sum_{n=0}^{N-1} e^{-2\pi i n \frac{1}{N} [k+l+m-N(N-1)/2 - \sum_\alpha r_\alpha]} \\
& \frac{1}{J(\lambda_1) J(\lambda_2)} \sum_{\beta, \gamma=1}^N \sum_{P, P'} (-1)^{P+P'} \exp \left[i \sum_{\alpha=0}^N \left(r_\alpha \lambda_1^{P(\alpha)} - (r_\alpha - l\delta_{\alpha\beta} - m\delta_{\alpha\gamma}) \lambda_2^{P'(\alpha)} \right) \right] \\
& \exp \left\{ -\frac{e^2 L \tau}{2} \left[\sum_{\alpha=1}^N r_\alpha^2 - \frac{1}{N} \left(\sum_{\alpha=1}^N r_\alpha \right)^2 \right] - \frac{e^2 L (\tau - u)}{2} \left[l^2 \left(\frac{N-1}{N} \right) + \frac{2l}{N} \sum_{\alpha=1}^N r_\alpha - 2l r_\beta \right] \right\} \\
& \exp \left\{ -\frac{e^2 L (\tau - v)}{2} \left[m^2 \left(\frac{N-1}{N} \right) + \frac{2m}{N} \sum_{\alpha=1}^N r_\alpha - 2m r_\gamma + 2lm \left(\delta_{\beta\gamma} - \frac{1}{N} \right) \right] \right\} , \quad (4.8)
\end{aligned}$$

the case $v < u$ can be obtained by exchanging $u \leftrightarrow v$. For $l = m = 0$ we get, as we should, 1. Even if in (4.8) the sum over the integers r_α is unrestricted, the presence of the determinant $\sum_P (-1)^P \exp(i \sum_\alpha r_\alpha \lambda_1^{P(\alpha)})$ forbids any couple of the r_α to take the same values. Consequently, one can easily reconstruct in (4.8) the second Casimir and the sum over representation as in Eq.(3.41). The sum will then be restricted only to those representations whose Young tableaux have rows with a number of boxes given by $k + l + m = 0, \text{ mod}[N]$ where k is the theta angle and l, m are the number of windings of the Wilson loops.

4.1 Wilson and Polyakov loop correlators on the torus

From Eq.(4.8) we can easily get the correlator of two Wilson (and, by duality $\tau \leftrightarrow L$, Polyakov) loops in the fundamental representation on the torus, just by taking the trace of Eq.(4.8) according to

$$P_{l,m}(\theta_k; u, v) = \int \prod_{\alpha=1}^N d\lambda_\alpha \delta\left(\sum_{\alpha=1}^N \lambda_\alpha\right) P_{l,m}(\theta_k, \lambda, \lambda; u, v). \quad (4.9)$$

Using the delta-function to eliminate the N^{th} component of λ , the integrations just give a product of Kronecker deltas imposing the conditions

$$r_\alpha - r_{P(\alpha)} + r_{P(N)} - r_N - l(\delta_{\alpha\beta} - \delta_{N\beta}) - m(\delta_{\alpha\gamma} - \delta_{N\gamma}) = 0 \quad \alpha = 1, \dots, N, \quad (4.10)$$

where the N^{th} equation does not arise from the integration but can be added because is automatically satisfied. The sum, from 1 to N , of these conditions gives

$$r_{P(N)} - r_N - \frac{1}{N}(m + l) + l\delta_{N\beta} + m\delta_{N\gamma} = 0, \quad (4.11)$$

consequently $m + l$ must be an integer multiple of N or must be zero otherwise the delta imposing the conditions (4.10) will set to zero the correlator. This is nothing but the Z_N invariance: all the correlators that are not Z_N -invariant must vanish. The correlator (4.9) is

$$\begin{aligned} P_{l,m}(\theta_k; u, v) &= \frac{N!}{N^2 Z_k} (2\pi)^{\frac{N-1}{2}} \sum_{\{r_1, \dots, r_{N-1}\}=-\infty}^{+\infty} \sum_{n=0}^{N-1} e^{-2\pi i n \frac{1}{N} [l+m+k-N(N-1)/2 - \sum_\alpha r_\alpha]} \\ &\sum_{\beta, \gamma=1}^N \sum_P (-1)^P \prod_{\alpha=1}^N \delta_{r_{P(\alpha)}, r_\alpha - l\delta_{\alpha\beta} - m\delta_{\alpha\gamma} + (m+l)/N} \\ &\exp \left\{ -\frac{e^2 L \tau}{2} \left[\sum_{\alpha=1}^N r_\alpha^2 - \frac{1}{N} \left(\sum_{\alpha=1}^N r_\alpha \right)^2 \right] - \frac{e^2 L (\tau - u)}{2} \left[l^2 \left(\frac{N-1}{N} \right) + \frac{2l}{N} \sum_{\alpha=1}^N r_\alpha - 2lr_\beta \right] \right\} \\ &\exp \left\{ -\frac{e^2 L (\tau - v)}{2} \left[m^2 \left(\frac{N-1}{N} \right) + \frac{2m}{N} \sum_{\alpha=1}^N r_\alpha - 2mr_\gamma + 2lm \left(\delta_{\beta\gamma} - \frac{1}{N} \right) \right] \right\}. \quad (4.12) \end{aligned}$$

Setting $m = 0$ from Eq.(4.12) one gets the correlator for a single loop. From Z_N invariance, this will be different from zero only if l is an integer multiple of N .

$$\begin{aligned} P_l(\theta_k; u) &= \frac{1}{N Z_k} \langle \text{Tr} \{ U^l(u) \} P_k \rangle = \frac{N!}{N Z_k} (2\pi)^{\frac{N-1}{2}} \cdot \\ &\sum_{\{r_1, \dots, r_{N-1}\}=-\infty}^{+\infty} \sum_{n=0}^{N-1} e^{-2\pi i n \frac{1}{N} [l+k-N(N-1)/2 - \sum_\alpha r_\alpha]} \sum_{\beta=1}^N \sum_P (-1)^P \prod_{\alpha=1}^N \delta_{r_{P(\alpha)}, r_\alpha - l\delta_{\alpha\beta} + l/N} \end{aligned}$$

$$\exp \left\{ -\frac{e^2 L \tau}{2} \left[\sum_{\alpha=1}^N r_{\alpha}^2 - \frac{1}{N} \left(\sum_{\alpha=1}^N r_{\alpha} \right)^2 \right] - \frac{2e^2 L (\tau - u)}{2} \left[l^2 \left(\frac{N-1}{N} \right) + \frac{2l}{N} \sum_{\alpha=1}^N r_{\alpha} - 2l r_{\beta} \right] \right\} . \quad (4.13)$$

For a loop anti-loop correlator, $m = -l$, Eq.(4.12) can be rewritten in a different form to make clear its invariance under translations on the torus

$$\begin{aligned} P_{l,-l}(\theta_k; u, v) &= \frac{(N-1)!}{Z_k} (2\pi)^{\frac{N-1}{2}} \sum_{\{r_1, \dots, r_{N-1}\}=-\infty}^{+\infty} \sum_{n=0}^{N-1} e^{-2\pi i n \frac{1}{N} [k - N(N-1)/2 - \sum_{\alpha} r_{\alpha}]} \\ &\left(\sum_{\beta=1}^N - \sum_{\beta, \gamma=1, \beta \neq \gamma}^N \delta_{r_{\beta}, r_{\gamma}-l} \right) \det_{i,j} \delta_{r_i, r_j} \exp \left\{ -\frac{e^2 L \tau}{2} \left[\sum_{\alpha=1}^N r_{\alpha}^2 - \frac{1}{N} \left(\sum_{\alpha=1}^N r_{\alpha} \right)^2 \right] \right\} \\ &\exp \left\{ -\frac{e^2 L |v - u|}{2} \left[l^2 \left(\frac{N-1}{N} \right) + \frac{2l}{N} \sum_{\alpha=1}^N r_{\alpha} - 2l r_{\beta} \right] \right\} . \end{aligned} \quad (4.14)$$

Eqs.(4.14) is different from the corresponding one obtained in Ref.[18]. As a matter of fact the restriction on the integers r_{α} introduced by the determinant factors, not only changes the spectrum of the theory but also alters the asymptotic behavior of the correlators. Consider for example the case of the correlator of a quark-antiquark pair, $l = 1$. As is known, any non-trivial theta-sector is unstable [32, 33, 34] in this case since the quark-antiquark pair is energetically favoured to reduce the associated background electric field. The asymptotic behavior of Eq.(4.14) can be obtained by sending $\tau \rightarrow \infty$. In this limit only the lowest representation survives in the sums in Eq.(4.14). The lowest value for the quantity $\sum_{\alpha} (r_{\alpha})^2 - (\sum_{\alpha} r_{\alpha})^2 / N$ can be obtained by choosing $r_{\alpha} = -\alpha$ (which corresponds to the singlet representation) and is given by the curvature of the group manifold $N(N^2 - 1)/12$ as in Eq.(3.42). The large τ limit of Eq.(4.14) is

$$\begin{aligned} \lim_{\tau \rightarrow \infty} P_{1,-1}(u, v) &= \left(\sum_{\beta=1}^N - \sum_{\beta, \gamma} \delta_{\beta, \gamma+1} \right) \exp \left[-\frac{e^2 L |v - u|}{2} \left(2\beta - \frac{1}{N} - N \right) \right] = \\ &= \exp \left(-\frac{e^2 L |v - u|}{2} \frac{N^2 - 1}{N} \right) \end{aligned} \quad (4.15)$$

We then have in this sector a confining potential between the quark-antiquark pair with string tension $e^2(N^2 - 1)/N$.

4.2 Results for $SU(2)$

Let us now study in particular the $SU(2)$ case on the torus. For $N = 2$ Eq.(4.14) reads

$$\begin{aligned} P_{l,-l}(\theta_k; u, v) &= \frac{2\sqrt{2}\pi}{Z_k} \left\{ \sum_{\{r \neq -2\}=-\infty}^{+\infty} \sum_{n=0}^1 e^{-\pi i n [k-1-r]} \right. \\ &\exp \left\{ -\frac{e^2 L \tau}{4} [(r+2)^2 + l^2 |v - u|] \right\} \cosh \left[\frac{e^2 L (v - u) l (r + 2)}{2} \right] \\ &\left. - \sum_{n=0}^1 e^{-\pi i n (k-1-l)} \exp \left\{ -\frac{e^2 L \tau l^2}{4} \left[1 - \frac{|v - u|}{\tau} \right] \right\} \right\} , \end{aligned} \quad (4.16)$$

Recall that the integers $r + 1$ for $r \geq -1$ or $-r - 3$ for $r \leq -3$, in Eq.(4.16) give the number of boxes in the rows of the Young table in a given representation. Consequently, for $k = 0$ only representations whose Young tableaux have an even number of boxes are present in the sum, for $k = 1$ only those with an odd number. Identifying $j = (r + 1)/2$ for $r \geq -1$ and $j = -(r + 3)/2$ for $r \leq -3$, $j = 0, 1/2, 1 \dots$ we can rewrite the loop correlator for $SU(2)$ as

$$P_{l,-l}(k=0; u, v) = \frac{2\sqrt{2\pi}}{Z_0} \left\{ 4 \sum_{\{j=0,1,\dots\}}^{+\infty} \exp \left\{ -e^2 L \tau \left[j(j+1) + \frac{1}{4} + \frac{l^2|v-u|}{4\tau} \right] \right\} \right. \\ \left. \cosh \left[\frac{e^2 L(v-u)l(2j+1)}{2} \right] - \sum_{n=0}^1 e^{\pi i n(1+l)} \exp \left\{ -\frac{e^2 L \tau l^2}{4} \left[1 - \frac{|v-u|}{\tau} \right] \right\} \right\} , \quad (4.17)$$

and

$$P_{l,-l}(k=1; u, v) = \frac{2\sqrt{2\pi}}{Z_1} \left\{ 4 \sum_{\{j=1/2,3/2,\dots\}}^{+\infty} \exp \left\{ -e^2 L \tau \left[j(j+1) + \frac{1}{4} + \frac{l^2|v-u|}{4\tau} \right] \right\} \right. \\ \left. \cosh \left[\frac{e^2 L(v-u)l(2j+1)}{2} \right] - \sum_{n=0}^1 e^{\pi i n l} \exp \left\{ -\frac{e^2 L \tau l^2}{4} \left[1 - \frac{|v-u|}{\tau} \right] \right\} \right\} , \quad (4.18)$$

Taking $\tau \rightarrow \infty$ in the fundamental representation with $l = 1$, we can see that the non trivial theta sector has an instability which creates a repulsive potential between the external charges. Moreover, the prescence of fundamental charges results breaks the center symmetry that gives rise to the theta-sectors in the first place. For example, with a pair of fundamental charges, one region of the torus carries fluxes with even numbers of boxes while another carries fluxes with odd numbers of boxes. In general one should sum over all theta-sectors in this case to obtain correct results for correlators. Turning to asymptotic behaviour, using Eq.(3.53) it is easy to see that the $\theta = 0$ sector produces the leading behaviour with inter-quark potential given by

$$\lim_{\tau \rightarrow \infty} -\frac{1}{L} \log P_{1,-1}(0; u, v) = \frac{3e^2}{4} |v - u| . \quad (4.19)$$

Let us consider now the correlator for a pair of adjoint $SU(2)$ loops. A loop in the adjoint representation can be taken as the modulus squared trace of the fundamental representation group element

$$\text{Tr} U_{\text{adj}}(u) = |\text{Tr} U(u)|^2 - 1 . \quad (4.20)$$

For $SU(2)$,

$$\text{Tr} U_{\text{adj}}(u) = \text{Tr} U^2(u) + 1 . \quad (4.21)$$

For the correlator we then have

$$\langle \text{Tr} U_{\text{adj}}(u) \text{Tr} U_{\text{adj}}(v) \rangle = 1 + \langle \text{Tr} U^2(u) \text{Tr} U^2(v) \rangle + \langle \text{Tr} U^2(u) \rangle + \langle \text{Tr} U^2(v) \rangle . \quad (4.22)$$

Using Eqs.(4.12) and (4.13)

$$\langle \text{Tr} U_{\text{adj}}(u) \text{Tr} U_{\text{adj}}(v) \rangle_k = 1 - 8 \frac{\sqrt{2\pi}}{Z_k} \sum_{n=0}^1 \exp \left[-\frac{e^2 L \tau}{4} - i\pi n k \right] \\ + 4 \frac{\sqrt{2\pi}}{Z_k} \left\{ \sum_{n=0}^1 \sum_{r=0}^{\infty} \exp \left[-\frac{e^2 L \tau (r+1)^2}{4} - i\pi n(k-r) \right] \left\{ \exp \left[e^2 L |v-u| r \right] \right. \right.$$

$$+ \exp \left[-e^2 L |v - u| (r + 2) \right] \Big\} - \sum_{n=0}^1 \exp \left[-e^2 L \tau + e^2 L |v - u| - i\pi n (k - 1) \right] \Big\} \quad (4.23)$$

As before, only representations with an even number of boxes, r even, will contribute when $k = 0$ and representations with an odd number of boxes when $k = 1$. Taking into account the behavior of Z_k , Eq.(3.53), for large τ , we obtain a confining behavior in the trivial theta-sector

$$\lim_{\tau \rightarrow \infty} -\frac{1}{L} \log \langle \text{Tr} U_{\text{adj}}(u) \text{Tr} U_{\text{adj}}(v) \rangle_0 = 2e^2 |v - u| \quad , \quad (4.24)$$

and a screening behavior in the non-trivial theta-sector.

$$\lim_{\tau \rightarrow \infty} -\frac{1}{L} \log \langle \text{Tr} U_{\text{adj}}(u) \text{Tr} U_{\text{adj}}(v) \rangle_1 = \frac{1}{L} \left[1 - \exp \left(-3e^2 L |v - u| \right) \right] \quad . \quad (4.25)$$

These results are in agreement with the general discussion of ref.[32].

As a check of previous calculations, we will now evaluate these correlators for the case of $SU(2)$, using the character representation. We start by considering the general case of the correlator of a pair of Polyakov loops in (irreducible) representations R and R' on a cylinder

$$P_{R,R'}(U_1, U_2; u, v) = \frac{1}{Z} \int [dV_1][dV_2] K[u; U_1, V_1] \chi_R(V_1) K[v - u; V_1, V_2] \chi_{R'}(V_2) K[\tau - v; V_2, U_2] \quad (4.26)$$

Using the definition of the kernel K (3.11) and the properties of the group characters this expression can be reduced to a sum over representations for $v > u$

$$P_{R,R'}(U_1, U_2; u, v) = \frac{1}{Z} \sum_{R_1, R_2, R_3} N_{RR_2}^{R_1} N_{R_3 R'}^{R_2} \chi_{R_1}(U_1) \chi_{R_3}(U_2) e^{-\{e^2 L [u C_2(R_1) + (v-u) C_2(R_2) + (\tau-v) C_2(R_3)]\}} \quad (4.27)$$

Using this result (4.27) we can easily find the analogous on the torus. We can set $U_1 = U_2 = U$ and integrate on U to obtain the pair correlator of Polyakov loops on the torus Using the properties of characters this reduces to

$$P_{R,R'}(u, v) = \frac{1}{Z} \sum_{R_1 R_2} N_{R_2 R}^{R_1} N_{R_1 R'}^{R_2} \exp \left(-e^2 L |v - u| C_2(R_1) - e^2 L (\tau - |u - v|) C_2(R_2) \right) \quad (4.28)$$

From this general formula one can immediately make quantitative statements about the binding between pairs of loops in the case where one side of the torus becomes large ($\tau \rightarrow \infty$). In the trivial theta-sector (recall this means we consider the sum of all distinct theta-sectors)

$$P_{R,R'}(u, v) \rightarrow \delta_{RR'} e^{-e^2 L |v - u| C_2(R)} \quad (4.29)$$

As well one can easily do the same for the correlator of a pair of Polyakov loops in the k^{th} theta-sector

$$P_{R,R'}(\theta_k; u, v) = \frac{1}{Z_k} \sum_{R_1 R_2} N_{R_2 R}^{R_1} N_{R_1 R'}^{R_2} \delta_N(k, C_1(R_1)) \exp \left(-e^2 L |v - u| C_2(R_1) - e^2 L (\tau - |v - u|) C_2(R_2) \right) \quad (4.30)$$

In the case of two Polyakov loops in the adjoint representation some calculation [33] shows for $N - 1 > k > 1$, the pair correlator is

$$P_{\text{Ad,Ad}}(\theta_k; u, v) \rightarrow (1 + e^{-e^2 L|v-u|k} + e^{-e^2 L|v-u|(N-k)} + e^{-e^2 L|v-u|(N+1)}) \quad (4.31)$$

The cases $k = 1$ and $k = N - 1$ are given by excluding the second and third terms respectively. The case $k = 0$ recovers the topologically trivial case (4.29) in this limit.

Let us now consider in particular the $SU(2)$ case where the fusion numbers are known explicitly. We will label representations by the single non-negative integer n which is equal to l_1 in terms of row variables or m_1 in terms of column variables in the associated Young table. Consequently, $C_2(n) = ((n + 1)^2 - 1)/4$. The fusion numbers are

$$N_{jl}^i = \begin{cases} 1 & \text{when } i = j + l, j + l - 2, \dots, |j - l| \\ 0 & \text{otherwise} \end{cases} \quad (4.32)$$

Bringing these facts together we have the correlator of pair of Polyakov loops, one in representation n and the other in m , separated by distance a $|v - u|$

$$P_{n,m}(u, v) = \frac{1}{Z} \sum_{j,l=0}^{\infty} e^{-\frac{e^2 L|v-u|}{4}[(l+1)^2-1]} e^{-\frac{e^2 L\tau}{4}[(j+1)^2-1]} \sum_{r=0}^{\min l,m} \sum_{s=0}^{\min j,s} \delta_{m+n,2(s+r)} \delta_{2l+2s+m,2j+2r+n} \quad (4.33)$$

The first delta function serves to enforce the condition that $m + n$ must be even. This is a particular example of the general fact in $SU(N)$ that in order for the correlator of any system of loops to be non-vanishing the total charge of the loops must be vanishing mod N . This restriction ensures that the system contains a charge singlet.

Two special cases that are of interest are the pair correlators of fundamental and adjoint Polyakov loops. For the fundamental case $n = m = 1$ and

$$P_{1,1}(u, v) = \frac{1}{Z} \sum_{l=0}^{\infty} e^{-\frac{e^2 L\tau}{4}[(l+1)^2-1]} \left[e^{-\frac{e^2 L|v-u|}{4}(2l+3)} + e^{-\frac{e^2 L(\tau-|v-u|)}{4}(2l+3)} \right] \quad (4.34)$$

It can be checked that this result coincides with the sum of Eqs. (4.17) (4.18) when the different zero point energy and normalization of Z in the two approaches, are taken into account. Likewise for the correlator of a pair of adjoint loops $m = n = 2$

$$P_{2,2}(u, v) = \frac{1}{Z} \left(e^{-2e^2 L|v-u|} + e^{-2e^2 L(\tau-|v-u|)} + \sum_{l=1}^{\infty} e^{-\frac{e^2 L\tau}{4}[(l+1)^2-1]} \left[e^{-e^2 L|v-u|(l+2)} + 1 + e^{-e^2 L(\tau-|v-u|)(l+2)} \right] \right) \quad (4.35)$$

Now we turn the issue of non-trivial θ -sectors in the correlators. As usual this is carried out by including a projection operator. For $SU(2)$ this prescription amounts to restricting the sum over representations to those with either odd or even numbers of boxes in the corresponding Young table. It should be noted that this can be done in a consistent manner only if each of the charges in the system has vanishing 2-ality (ie. has $2n$ boxes).

For the correlator of a pair of adjoint Polyakov loops the two cases are $k = 0$ where one would sum over even representations in (4.33)

$$P_{2,2}(\theta_0; u, v) = \frac{1}{Z_0} \left(e^{-2e^2 L|v-u|} + e^{-2e^2 L(\tau-|v-u|)} \right) \quad (4.36)$$

$$+ \sum_{l=2,4,\dots}^{\infty} e^{-\frac{e^2 L \tau}{4}[(l+1)^2-1]} \left[e^{-e^2 L|v-u|(l+2)} + 1 + e^{-e^2 L(\tau-|v-u|)(l+2)} \right] \quad (4.37)$$

and the case $k = 1$ where the sum is over odd representations

$$P_{2,2}(\theta_1; u, v) = \frac{1}{Z_1} \sum_{l=1,3,\dots}^{\infty} e^{-\frac{e^2 L \tau}{4}[(l+1)^2-1]} \left[e^{-e^2 L|u-v|(l+2)} + 1 + e^{-e^2 L(\tau-|v-u|)(l+2)} \right] \quad (4.38)$$

These results coincide with the results of the functional integral technique, Eq.(4.23).

5 Discussion

We have demonstrated the utility of the diagonalization method for doing practical computations in 2D Yang-Mills theory. This method is alternative to the well-known character expansion technique. The diagonalization method is useful for computing correlators of higher winding loop operators. Knowing the formulae for higher representations in terms of higher winding loops (via characters), and the formulae for correlators in the character representation, it should be possible to obtain information about the fusion rules for higher representations of SU(N) groups (and in fact, by suitable generalization of our work, arbitrary Lie groups) from our formulae. We have not pursued this problem in the present work.

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